

Two-dimensional Systems that Arise from the Noether Classification of Lagrangians on the Line

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Abstract

Noether-like operators play an essential role in writing down the first integrals for Euler-Lagrange systems of ordinary differential equations (ODEs). The classification of such operators is carried out with the help of analytic continuation of Lagrangians on the line. We obtain the classification of 5, 6 and 9 Noether-like operators for two-dimensional Lagrangian systems that arise from the submaximal and maximal dimensional Noether point symmetry classification of Lagrangians on the line. Cases in which the Noether-like operators are also Noether point symmetries for the systems of two ODEs are mentioned. In particular, the 8-dimensional maximal Noether algebra is remarkably obtained for the simplest system of the free particle equations in two dimensions from the 5-dimensional complex Noether algebra of the standard Lagrangian of the scalar free particle equation. We present the effectiveness of Noether-like operators for the determination of first integrals of systems of two nonlinear differential equations which arise from scalar complex Euler-Lagrange ODEs that admit Noether symmetry.

Key words: Complex Lagrangian, Noether-like operators, Noether classification.

1 Introduction

Over the last few decades there have been many contributions to the study of Lie and Noether point symmetries for second-order ODEs. The use of these symmetries signifies their importance in the reduction of order of a

given dynamical equation and in constructing its first integrals (constants of the motion). Lie proved that the maximum dimension of the point symmetry algebra for a scalar second-order ODE is eight. He showed that the simple differential equation $u'' = 0$, admits the maximal symmetry algebra which corresponds to the Lie algebra $sl(3, \mathbb{R})$ and there exists a linearizing point transformation for those second-order scalar ODEs that possess this algebra. Further, any other algebra admitted by a scalar second-order ODE is a sub-algebra of $sl(3, \mathbb{R})$ (see [1]). Lie also proved that a scalar second-order ODE admits the maximal $r \in \{0, 1, 2, 3, 8\}$ –dimensional point symmetry algebras [2] (see also [1]). Moreover, it is also known that the maximum Lie algebra of point symmetries admitted by a system of two second-order ODEs is fifteen [3].

There are at least two important facets in Lagrangian mechanics. One is to find a Lagrangian and the other is to construct first integrals of the underlying Euler-Lagrange (EL) differential equations. The classical Noether theorem requires the presence of a Lagrangian for a differential equation before its first integral can be evaluated with the help of an explicit formula [4]. The first integrals not only play a vital role in the integrability of an ODE but also have direct physical consequences. Douglas [5] gave the solution to the inverse problem for a system of two second-order ODEs. It is at times very difficult to find a Lagrangian and there are differential equations (see, e.g., Anderson and Thompson [6]) that do not admit Lagrangians. It raises another important question: can we find first integrals without a variational structure? The construction of conservation laws in the absence of a Lagrangian is carried out by Kara and Mahomed [7]. Ibragimov [8], also discussed a way of finding conservation laws without prior knowledge of a Lagrangian. In [9], Kara and Mahomed introduced the partial Noether approach to construct conserved quantities. They derived first integrals for those differential equations that fail to admit Lagrangians. Furthermore for nonvariational differential equations the Noether approach is also used to find first integrals recently by Gouveia and Torres in [10]. Moreover the proof of more general forms of the Noether theorem and conservation laws is a subject under strong development see, e.g., [11,12,13].

The search for new conservation laws for those systems of two ODEs that appear as dynamical equations in mathematical physics is indispensable. In [14], Gorringer and Leach classified the Lie algebra for systems of two second-order ODEs with constant coefficients. Later, Wafo and Mahomed discussed the case with variable coefficients in [15]. The classification of Noether point symmetries of an autonomous quadratic in the velocities Lagrangian with two degrees of freedom is done by Sen [16]. We know that the maximum dimension of the Noether point symmetry algebra for a Lagrangian in one-dimensional particle dynamics is five. A first-order Lagrangian on the line can have Noether point symmetry algebras of dimension 0, 1, 2, 3 or 5 [17,18,19].

Besides, a two dimensional system of free particle equations has 8–dimensional Noether algebra [3,20]. The classification of Noether subalgebras for systems of two nonlinear EL ODEs has not been carried out before. Our aim is to classify Noether operators for two-dimensional systems that arise from the complex Noether point symmetry classification of complex Lagrangians on the line. We make use of Noether-like operators for such nonlinear systems by utilizing complex arguments.

In [21,22,23], the authors used analytic continuation of ODEs in the complex plane to obtain non-trivial results for systems of differential equations. In particular, symmetry analysis on the complex plane leads to finding the symmetries, reduction of order, linearization and conservation laws of systems of two real ODEs and PDEs. In this paper, we furnish important results on Noether operators and the production of first integrals of systems of two second-order ODEs. It is mainly done by confining the domain of definition of dependent functions on a single real line and then the dynamics is studied on the complex plane. We employ a procedure which is similar to analytic continuation in the intervening steps but require that the dependent variables may be restricted on the line to achieve operators and first integrals of systems of ODEs. To achieve our goal we start with restricted complex ordinary differential equations (r-CODEs) [21]. Such r-CODEs are obtained by allowing a complex function to depend only on a single real variable which thereby yields a system of two ODEs. Our aim is thus to investigate the algebraic properties of systems of EL ODEs encoded in r-CODEs that admit Lagrangians. Indeed, it is shown that a complex Lagrangian reveals two real inequivalent Lagrangians for systems of two ODEs, i.e., these do not differ by a divergence. Thus, the classification of Lagrangians on the line would offer us a great deal of information about the inverse problem, algebraic properties and first integrals of the corresponding systems of two ODEs. In this regard, we mention that the complex extension of Hamiltonians and Lagrangians has been discussed by Bender [24]. He proved various intriguing results that verified known facts about quantum mechanics. The essence of his approach lies in the fact that a complex Hamiltonian can also be taken for a consistent physical theory of quantum mechanics.

Unlike a complex Lagrangian, a complex symmetry of an r-CODE may not give two real symmetries of the system, in general. It splits into two operators. We call such operators *Lie-like operators*. These operators may not necessarily be symmetries of the systems of ODEs corresponding to an r-CODE. Further, these operators do not form an algebra in general. Similarly, we call *Noether-like operators* those operators that are obtainable from a complex Noether symmetry. To draw attention on the significance of these operators, a few examples are discussed in some detail. The complex Noether theorem enables a description of an explicit formula for Noether-like operators. It is shown explicitly that the first integrals for such systems of two second-order ODEs

corresponding to these operators associated with the Lagrangians can be determined. An appealing consequence emerges when a single real symmetry permits the existence of two first integrals for the corresponding system.

We obtain a classification of a system of two EL ODEs with respect to the Noether-like operators they admit. We see that systems of ODEs that exhibit the same invariance properties as r-CODEs have the same structure of the Noether-like operators. We construct an analogue of the Noether counting theorem for systems of two second-order EL ODEs that arise from the Lagrangian formulation of scalar second-order equations. We obtain 5 and 6 Noether-like operators for systems of two second-order ODEs by the analytic continuation of the 3-dimensional Noether algebra of an r-CODE. The Noether algebra of a restricted complexified free particle equation which is 5-dimensional remarkably generates an 8-dimensional Noether algebra of the simplest system which exhibits a non-trivial implication of the complex variable approach. Moreover, *nine* Noether-like operators imply *ten* first integrals whereas in the classical Noether approach, there are only eight first integrals corresponding to eight Noether symmetries. This indeed is a nice result which we derive here. It is also conjectured that all linear systems of two second-order ODEs that can be derived from a complex variational principal of a scalar linear second-order ODE, indeed, admit an eight dimensional Noether algebra.

The outline of the paper is as follows. In the next section, we present the preliminaries. The EL equations are obtained and the Noether theorem is invoked in order to write down the Noether-like symmetry conditions for systems of two ODEs. An explicit formula to find first integrals corresponding to Noether-like operators is also mentioned for such systems. Few cases are described that shed light on the comparison of two formulae for acquiring first integrals for systems of ODEs. The classification of 5 and 6 Noether-like operators is carried out in the third section. The second last section deals with the case of maximal Noether symmetry algebra that is attained via complex variables. Some physical insights are also developed in the same section. Finally, we conclude the discussion in the last section.

2 Preliminaries

The problem of our interest is to study and classify algebraic properties and invariants for systems of two second-order ODEs of the form

$$\begin{aligned} f'' &= w_1(x, f, g, f', g'), \\ g'' &= w_2(x, f, g, f', g'), \end{aligned} \tag{1}$$

by invoking a complex analytic structure on the fg -plane. This study helps in discovering conserved quantities for those systems of ODEs that appear in diverse physical phenomena. For example, in the case of two coupled nonlinear oscillators the governing dynamical equations have the form of (1). There has been considerable amount of work done in finding all the conserved quantities of time dependent and independent nonlinear oscillators. The search for new conservation laws for nonlinear systems is one of the main objectives of the physicists. We assume a variational structure on the system (1). We show how the Noether point symmetry classification of complex Lagrangians on the line guarantees the emergence of operators and invariants for a class of systems of the form (1). It is emphasized that the use of complex Lagrangians is inevitable for such systems.

We assume that the above system has a complex structure, i.e., there exists a transformation

$$u(x) = f(x) + ig(x), \quad (2)$$

that maps system (1) to

$$u'' = w(x, u, u'), \quad (3)$$

which is a second-order r-CODE. It is clear that an arbitrary system (1) may or may not necessarily correspond to an r-CODE. Here we are interested in looking for the insights that can be extracted for systems of two ODEs (1) from equations of the form (3). The symmetry analysis of such systems (1) is carried out in [21,22,23]. The complex Lie algebra of (3) gives the real Lie-like operators of the system (1) [21,22,23]. We intend to investigate the implications of the variational structure on such systems of differential equations via the complex plane. The following details are not in conflict with conventional symmetry analysis but is rather a natural generalization of it in the complex domain. The equation (3) may be regarded as an analytic continuation of a general scalar second-order ODE in the restricted complex domain, i.e., here u is a complex function of a real variable x .

Suppose that the equation (3) appears from a variational principle, i.e., there exists a complex Lagrangian $L(x, u, u')$ such that the EL equation implies (3). Once the Lagrangian of a differential equation is known, its symmetry properties are explored. In the subsequent discussions we consider EL equations, Noether-like symmetry conditions and formulae of first integrals for the systems of two ODEs.

Theorem. *If $L(x, u, u') = L_1 + iL_2$, is a complex Lagrangian of a second-order r-CODE then both $L_1(x, f, g, f', g')$ and $L_2(x, f, g, f', g')$ are two Lagrangians of the corresponding system of two second-order ODEs.*

Proof. Suppose that L is a complex Lagrangian of the r-CODE (3) relative to system (1). Therefore, it satisfies the complex EL equation. The realification of the EL equation yields

$$\begin{aligned}\frac{\partial L_1}{\partial f} + \frac{\partial L_2}{\partial g} - \frac{d}{dx} \left(\frac{\partial L_1}{\partial f'} + \frac{\partial L_2}{\partial g'} \right) &= 0, \\ \frac{\partial L_2}{\partial f} - \frac{\partial L_1}{\partial g} - \frac{d}{dx} \left(\frac{\partial L_2}{\partial f'} - \frac{\partial L_1}{\partial g'} \right) &= 0.\end{aligned}\tag{4}$$

Since $L(x, u, u')$ is complex analytic in its arguments, both L_1 and L_2 satisfy the Cauchy-Riemann equations and the above system becomes

$$\begin{aligned}\frac{\partial L_1}{\partial f} - \frac{d}{dx} \left(\frac{\partial L_1}{\partial f'} \right) &= 0, \quad \frac{\partial L_1}{\partial g} - \frac{d}{dx} \left(\frac{\partial L_1}{\partial g'} \right) = 0, \\ \frac{\partial L_2}{\partial f} - \frac{d}{dx} \left(\frac{\partial L_2}{\partial f'} \right) &= 0, \quad \frac{\partial L_2}{\partial g} - \frac{d}{dx} \left(\frac{\partial L_2}{\partial g'} \right) = 0.\end{aligned}\tag{5}$$

The above equations are the usual EL-equations for the system (1). Hence L_1 and L_2 are two Lagrangians for the system (1).

Notice that these two Lagrangians do not differ by a divergence. It would be nice to deduce the symmetry properties of these Lagrangians via the complex Lagrangian. We would come to this point later as well as in the discussion of alternative Lagrangians.

Definition. The operators $\mathbf{X}_1 = 2\varsigma_1 \partial_x + \chi_1 \partial_f + \chi_2 \partial_g$ and $\mathbf{X}_2 = 2\varsigma_2 \partial_x + \chi_2 \partial_f - \chi_1 \partial_g$ are said to be Noether-like operators of system (1) with respect to the Lagrangians L_1 and L_2 if they satisfy

$$\begin{aligned}\mathbf{X}_1^{(1)} L_1 - \mathbf{X}_2^{(1)} L_2 + (d_x \varsigma_1) L_1 - (d_x \varsigma_2) L_2 &= d_x A_1, \quad d_x = d/dx, \\ \mathbf{X}_1^{(1)} L_2 + \mathbf{X}_2^{(1)} L_1 + (d_x \varsigma_1) L_2 + (d_x \varsigma_2) L_1 &= d_x A_2,\end{aligned}\tag{6}$$

for suitable functions A_1 and A_2 .

We now set

$$\varsigma = \varsigma_1 + i\varsigma_2, \quad A = A_1 + iA_2, \quad \mathbf{Z} = \mathbf{X}_1 + i\mathbf{X}_2.\tag{7}$$

Further, if we let $\chi = \chi_1 + i\chi_2$ and $\chi^{(1)} = \chi_1^{(1)} + i\chi_2^{(1)}$ in the complex symmetry

$$\mathbf{Z}^{(1)} = \varsigma \frac{\partial}{\partial x} + \chi \frac{\partial}{\partial u} + \chi^{(1)} \frac{\partial}{\partial u'},\tag{8}$$

then $\mathbf{X}_1^{(1)}$ and $\mathbf{X}_2^{(1)}$ are

$$\begin{aligned}\mathbf{X}_1^{(1)} &= 2\varsigma_1\partial_x + \chi_1\partial_f + \chi_2\partial_g + \chi_1^{(1)}\partial_{f'} + \chi_2^{(1)}\partial_{g'}, \\ \mathbf{X}_2^{(1)} &= 2\varsigma_2\partial_x + \chi_2\partial_f - \chi_1\partial_g + \chi_2^{(1)}\partial_{f'} - \chi_1^{(1)}\partial_{g'}.\end{aligned}\tag{9}$$

We state these as the first prolongations of the Noether-like operators \mathbf{X}_1 and \mathbf{X}_2 .

The above conditions are different from the usual Noether conditions for the systems of two ODEs. These operators do not form an algebra in general. It may be questioned as to what is the use of these operators. The point is that we can determine invariants by employing such operators. To provide concrete basis to our argument, we mention two cases in which the conditions (6) reduce to the usual Noether conditions: (a) if \mathbf{Z} has either a pure real or pure imaginary form then \mathbf{Z} becomes a Noether symmetry for both Lagrangians. For example $\mathbf{Z} = \mathbf{X}_1$, i.e., it has only a real part so (6) takes the form

$$\begin{aligned}\mathbf{X}_1^{(1)}L_1 + (d_x\varsigma_1)L_1 &= d_xA_1, \\ \mathbf{X}_1^{(1)}L_2 + (d_x\varsigma_1)L_2 &= d_xA_2,\end{aligned}\tag{10}$$

and hence \mathbf{X}_1 is a Noether symmetry surprisingly for the system (1) relative to both inequivalent Lagrangians L_1 and L_2 , (b) if $L = L_1$ then (6) becomes

$$\begin{aligned}\mathbf{X}_1^{(1)}L_1 + (d_x\varsigma_1)L_1 &= d_xA_1, \\ \mathbf{X}_2^{(1)}L_1 + (d_x\varsigma_2)L_1 &= d_xA_2,\end{aligned}\tag{11}$$

in which case \mathbf{X}_1 and \mathbf{X}_2 turn out to be two distinct Noether symmetries for the system (1) corresponding to L_1 .

Noether-like Theorem. *If \mathbf{X}_1 and \mathbf{X}_2 are two Noether-like operators of (1) with respect to the Lagrangians L_1 and L_2 then (1) admits two first integrals*

$$\begin{aligned}I_1 &= \varsigma_1L_1 - \varsigma_2L_2 + \partial_{f'}L_1(\chi_1 - f'\varsigma_1 - g'\varsigma_2) - \partial_{f'}L_2(\chi_2 - f'\varsigma_2 - g'\varsigma_1) - A_1, \\ I_2 &= \varsigma_1L_2 + \varsigma_2L_1 + \partial_{f'}L_2(\chi_1 - f'\varsigma_1 - g'\varsigma_2) + \partial_{f'}L_1(\chi_2 - f'\varsigma_2 - g'\varsigma_1) - A_2.\end{aligned}\tag{12}$$

Proof. That I_1 and I_2 are two first integrals of system (1) can be verified by

$$d_xI_1 = 0, \quad d_xI_2 = 0.\tag{13}$$

on the system (1).

Notice that formulae (12) for the first integrals are different from the usual Noether first integrals of a system. We can find the usual formulae for the above mentioned cases (a) and (b) as follows. In case (a), we get

$$\begin{aligned}
I_1 &= \varsigma_1 L_1 - f' \varsigma_1 (\partial_{f'} L_1) + g' \varsigma_1 (\partial_{g'} L_2) - A_1, \\
I_2 &= \varsigma_1 L_2 - f' \varsigma_1 (\partial_{f'} L_2) - g' \varsigma_1 (\partial_{g'} L_1) - A_2.
\end{aligned} \tag{14}$$

It may seem strange that we are *obtaining* two first integrals for (1) corresponding to a single symmetry \mathbf{X}_1 (or \mathbf{X}_2). The point is that \mathbf{X}_1 (or \mathbf{X}_2) is the Noether symmetry of both L_1 and L_2 . A simple question arises as to which of the Lagrangians results in maximum number of Noether symmetries. The answer helps in writing all the first integrals of systems of two ODEs corresponding to such Noether symmetries. It is quite natural to see that in case (b) we get two real first integrals corresponding to \mathbf{X}_1 and \mathbf{X}_2 associated with L_1 , i.e.,

$$\begin{aligned}
I_1 &= \varsigma_1 L_1 + (\chi_1 - f' \varsigma_1 - g' \varsigma_2) \partial_{f'} L_1 + (\chi_2 - f' \varsigma_2 - g' \varsigma_1) \partial_{g'} L_1 - A_1, \\
I_2 &= \varsigma_2 L_1 - (\chi_1 - f' \varsigma_1 - g' \varsigma_2) \partial_{g'} L_1 + (\chi_2 - f' \varsigma_2 - g' \varsigma_1) \partial_{f'} L_1 - A_2.
\end{aligned} \tag{15}$$

The two first integrals satisfy the coupled equations

$$\begin{aligned}
\mathbf{X}_1^{(1)} I_1 - \mathbf{X}_2^{(1)} I_2 &= 0, \\
\mathbf{X}_1^{(1)} I_2 + \mathbf{X}_2^{(1)} I_1 &= 0,
\end{aligned} \tag{16}$$

where $\mathbf{Z}^{(1)} = \mathbf{X}_1^{(1)} + i\mathbf{X}_2^{(1)}$.

To illustrate we commence with those systems of second-order ODEs that admit 1, 2, 3 or 4 Noether-like operators. In order to carry out calculations we use Computer Algebra System (CAS), e.g., MAPLE and CRACK [25,26]. The first two examples briefly explain the determination of these operators and first integrals. It is noticed that these operators could be supplied by the analytical continuation of 2-dimensional Noether algebras. There exists two types of realizations of both one and two dimensional complex algebras [2,19]. The one dimensional realization includes $\partial/\partial x$ and $\partial/\partial u$ that correspond to a single and two Noether-like operators. Similarly, $\{\partial/\partial x, \partial/\partial u\}$ and $\{\partial/\partial u, x\partial/\partial x + u\partial/\partial u\}$ are amongst the realizations of two dimensional algebras in the complex domain.

Applications:

1. In this example, we show how Noether-like operators are used to construct invariants for the systems. We start with the system of two second-order non-linear ODEs

$$\begin{aligned}
ff'' - gg'' &= e^{-f'} \cos(g'), \\
fg'' + gf'' &= -e^{-f'} \sin(g').
\end{aligned} \tag{17}$$

The above equations are equivalent to a system of EL equations (4) on employing the Lagrangians

$$L_1 = e^{f'} \cos(g') + \frac{1}{2} \ln(f^2 + g^2), \quad L_2 = e^{f'} \sin(g') + \arctan\left(\frac{g}{f}\right). \quad (18)$$

Now the Noether-like symmetry conditions (6) for the system of ODEs has the form

$$\begin{aligned} A_{1x} + f'(A_{1f} + A_{2g}) - g'(A_{2f} - A_{1g}) &= \frac{\chi_{1f} + \chi_{2g}}{f^2 + g^2} + \\ &e^{f'} [\{\chi_{1x} + \varsigma_{1x} + f'(\chi_{1f} + \chi_{2g} - \varsigma_{1x} + \varsigma_{1f} + \varsigma_{2g}) - \\ &g'(\chi_{2f} - \chi_{1g} - \varsigma_{2x} + \varsigma_{2f} - \varsigma_{1g}) - (f'^2 - g'^2)(\varsigma_{1f} + \varsigma_{2g}) + \\ &2f'g'(\varsigma_{2f} - \varsigma_{1g})\} \cos(g') - \{\chi_{2x} + \varsigma_{2x} + g'(\chi_{1f} + \chi_{2g} - \\ &\varsigma_{1x} + \varsigma_{1f} + \varsigma_{2g}) + f'(\chi_{2f} - \chi_{1g} - \varsigma_{2x} + \varsigma_{2f} - \varsigma_{1g}) - \\ &2f'g'(\varsigma_{2f} - \varsigma_{1g}) - (f'^2 - g'^2)(\varsigma_{1f} + \varsigma_{2g})\} \sin(g')] + \\ &\frac{1}{2} \ln(f^2 + g^2) [\varsigma_{1x} + f'(\varsigma_{1f} + \varsigma_{2g}) - g'(\varsigma_{2f} - \varsigma_{1g})] - \\ &\arctan(g/f) [\varsigma_{2x} + g'(\varsigma_{1f} + \varsigma_{2g}) + f'(\varsigma_{2f} - \varsigma_{1g})], \end{aligned} \quad (19)$$

$$\begin{aligned} A_{2x} + f'(A_{2f} - A_{1g}) + g'(A_{1f} + A_{2g}) &= \frac{\chi_{2f} - \chi_{1g}}{f^2 + g^2} + \\ &e^{f'} [\{\chi_{1x} + \varsigma_{1x} + f'(\chi_{1f} + \chi_{2g} - \varsigma_{1x} + \varsigma_{1f} + \varsigma_{2g}) - \\ &g'(\chi_{2f} - \chi_{1g} - \varsigma_{2x} + \varsigma_{2f} - \varsigma_{1g}) - (f'^2 - g'^2)(\varsigma_{1f} + \varsigma_{2g}) + \\ &2f'g'(\varsigma_{2f} - \varsigma_{1g})\} \sin(g') + \{\chi_{2x} + \varsigma_{2x} + g'(\chi_{1f} + \chi_{2g} - \\ &\varsigma_{1x} + \varsigma_{1f} + \varsigma_{2g}) + f'(\chi_{2f} - \chi_{1g} - \varsigma_{2x} + \varsigma_{2f} - \varsigma_{1g}) - \\ &2f'g'(\varsigma_{2f} - \varsigma_{1g}) - (f'^2 - g'^2)(\varsigma_{1f} + \varsigma_{2g})\} \cos(g')] + \\ &\arctan(g/f) [\varsigma_{1x} + f'(\varsigma_{1f} + \varsigma_{2g}) - g'(\varsigma_{2f} - \varsigma_{1g})] + \\ &\frac{1}{2} \ln(f^2 + g^2) [\varsigma_{2x} + g'(\varsigma_{1f} + \varsigma_{2g}) + f'(\varsigma_{2f} - \varsigma_{1g})]. \end{aligned} \quad (20)$$

On comparing the coefficients of all independent quantities in the above equations, we get the following system of linear PDEs

$$\varsigma_{1f} + \varsigma_{2g} = 0, \quad \varsigma_{2f} - \varsigma_{1g} = 0, \quad (21)$$

$$\chi_{1x} + \varsigma_{1x} = 0, \quad \chi_{2x} + \varsigma_{2x} = 0, \quad (22)$$

$$\chi_{1f} + \chi_{2g} - \varsigma_{1x} = 0, \quad (23)$$

$$\chi_{2f} - \chi_{1g} - \varsigma_{2x} = 0, \quad (24)$$

$$A_{1f} + A_{2g} = 0, \quad A_{2f} - A_{1g} = 0, \quad (25)$$

$$\begin{aligned} \frac{f\chi_1 + g\chi_2}{f^2 + g^2} + \frac{1}{2} \ln(f^2 + g^2) \varsigma_{1x} - \arctan(g/f) \varsigma_{2x} &= A_{1x}, \\ \frac{f\chi_2 - g\chi_1}{f^2 + g^2} + \frac{1}{2} \ln(f^2 + g^2) \varsigma_{2x} + \arctan(g/f) \varsigma_{1x} &= A_{2x}. \end{aligned} \quad (26)$$

Eqs. (21) and (25) upon using the analyticity of ς_1 , ς_2 , A_1 and A_2 imply that

$$\varsigma_1 = \varsigma_1(x), \quad \varsigma_2 = \varsigma_2(x), \quad A_1 = A_1(x), \quad A_2 = A_2(x).$$

Eqs. (23) and (24) on utilizing above yield

$$\chi_1 = f\varsigma'_1 - g\varsigma'_2 + g_1(x), \quad \chi_2 = f\varsigma'_2 + g\varsigma'_1 + g_2(x). \quad (27)$$

Insertion of (27) in (22) gives

$$\chi_1 = (f - x)C_1 - gC_2, \quad \chi_2 = (f - x)C_2 + gC_1. \quad (28)$$

Now by differentiating (26) with respect of f and g and using Eqs. (28) in it gives us the following solution $C_1 = \chi_1 = 0 = \chi_2 = C_2$, and $\varsigma_1 = C_3$, $\varsigma_2 = C_4$, while the gauge functions are determined to be $A_1 = C_5$, $A_2 = C_6$. Therefore, we obtain a single Noether-like operator which is translation in x of system (17). Hence, by utilizing (12) the invariants of (17) are

$$\begin{aligned} I_1 &= e^{f'} \cos(g') + \frac{1}{2} \ln(f^2 + g^2) - e^{f'} (f' \cos(g') - g' \sin(g')), \\ I_2 &= e^{f'} \sin(g') + \arctan(g/f) - e^{f'} (f' \sin(g') + g' \cos(g')). \end{aligned} \quad (29)$$

This case corresponds to the one-dimensional realization of the complex algebra spanned by ∂_x mentioned above. In fact system (17) can be converted into an r-CODE $uu'' = e^{-u'}$, which has Lagrangian $L = e^{u'} + \log u$.

2. In this example, we study the invariant properties of the system

$$\begin{aligned} 2f'' + f'f'' - g'g'' &= xe^{-f'} \cos(g'), \\ 2g'' + f'g'' + g'f'' &= -xe^{-f'} \sin(g'). \end{aligned} \quad (30)$$

It can be checked that the operators ∂_f and ∂_g satisfy (6) which thus provide two Noether-like operators for the above system relative to

$$\begin{aligned} L_1 &= e^{f'} (f' \cos(g') - g' \sin(g')) + xf, \\ L_2 &= e^{f'} (f' \sin(g') + g' \cos(g')) + xg, \end{aligned} \quad (31)$$

by solving Noether-like symmetry conditions (6)

$$\begin{aligned} A_{1x} + f'(A_{1f} + A_{2g}) - g'(A_{2f} - A_{1g}) &= f\varsigma_1 - g\varsigma_2 + \chi_1 x + [\{\chi_{1x} + \\ &f'(\chi_{1f} + \chi_{2g}) - g'(\chi_{2f} - \chi_{1g}) - f'(\varsigma_{1f} + \varsigma_{2g}) + g'(\varsigma_{2f} - \varsigma_{1g}) - \\ &(f'^2 - g'^2)(\varsigma_{1f} + \varsigma_{2g}) + 2f'g'(\varsigma_{2f} - \varsigma_{1g})\}e^{f'}(\cos g'(1 + f') - g' \sin g')] \end{aligned}$$

$$\begin{aligned}
& -\{\chi_{2x} + g'(\chi_{1f} + \chi_{2g}) + f'(\chi_{2f} - \chi_{1g}) + f'(\varsigma_{2f} - \varsigma_{1g}) - g'(\varsigma_{1f} + \varsigma_{2g}) - \\
& 2f'g'(\varsigma_{1f} - \varsigma_{2g}) - (f'^2 - g'^2)(\varsigma_{2f} - \varsigma_{1g})\}e^{f'}((1 + gf')\sin g' + g'\cos g') + \\
& x(f\varsigma_{1x} - g\varsigma_{2x}) + x(ff' - gg')(\varsigma_{1f} + \varsigma_{2g}) - x(fg' + f'g)(\varsigma_{2f} - \varsigma_{1g}) \\
& e^{f'}[(f'\cos g' - g'\sin g')\varsigma_{1x} - (f'\sin g' + g'\cos g')\varsigma_{2x} + ((f'^2 - g'^2)\cos g' - \\
& 2f'g'\sin g')(\varsigma_{1f} + \varsigma_{2g}) - ((f'^2 - g'^2)\sin g' + 2f'g'\cos g')(\varsigma_{2f} - \varsigma_{1g})].
\end{aligned} \tag{32}$$

$$\begin{aligned}
& A_{2x} + g'(A_{1f} + A_{2g}) + f'(A_{2f} - A_{1g}) = g\varsigma_1 + f\varsigma_2 + \chi_{2x} + [\{\chi_{1x} + \\
& f'(\chi_{1f} + \chi_{2g}) - g'(\chi_{2f} - \chi_{1g}) - f'(\varsigma_{1f} + \varsigma_{2g}) + g'(\varsigma_{2f} - \varsigma_{1g}) - \\
& (f'^2 - g'^2)(\varsigma_{1f} + \varsigma_{2g}) + 2f'g'(\varsigma_{2f} - \varsigma_{1g})\}e^{f'}((1 + f')\sin g' + g'\cos g') \\
& + \{\chi_{2x} + g'(\chi_{1f} + \chi_{2g}) + f'(\chi_{2f} - \chi_{1g}) + f'(\varsigma_{2f} - \varsigma_{1g}) - g'(\varsigma_{1f} + \varsigma_{2g}) - \\
& 2f'g'(\varsigma_{1f} + \varsigma_{2g}) - (f'^2 - g'^2)(\varsigma_{2f} - \varsigma_{1g})\}e^{f'}((1 + f')\cos g' - g'\sin g')] + \\
& x(g\varsigma_{1x} + f\varsigma_{2x}) + x(ff' - gg')(\varsigma_{2f} - \varsigma_{1g}) + x(fg' + f'g)(\varsigma_{1f} + \varsigma_{2g}) \\
& e^{f'}[(f'\cos g' - g'\sin g')\varsigma_{2x} + (f'\sin g' + g'\cos g')\varsigma_{1x} \\
& + (f'^2 - g'^2)(\sin g'(\varsigma_{1f} + \varsigma_{2g}) + \cos g'(\varsigma_{2f} - \varsigma_{1g})) \\
& + 2f'g'(\cos g'(\varsigma_{1f} + \varsigma_{2g}) - \sin g'(\varsigma_{2f} - \varsigma_{1g}))].
\end{aligned} \tag{33}$$

Again, by comparison of the coefficients of independent functions, the above equations give rise to

$$\varsigma_{1x} = 0, \quad \varsigma_{2x} = 0, \tag{34}$$

$$\varsigma_{1f} + \varsigma_{2g} = 0, \quad \varsigma_{2f} - \varsigma_{1g} = 0, \tag{35}$$

$$\chi_{1x} = 0, \quad \chi_{2x} = 0, \tag{36}$$

$$\chi_{1f} + \chi_{2g} = 0, \quad \chi_{2f} - \chi_{1g} = 0, \tag{37}$$

$$\varsigma_{1f} - \varsigma_{2g} + \chi_{1x} = A_{1x}, \tag{38}$$

$$\varsigma_{1g} + \varsigma_{2f} + \chi_{2x} = A_{2x}, \tag{39}$$

$$A_{1f} + A_{2g} = 0, \quad A_{2f} - A_{1g} = 0. \tag{40}$$

Equations (34)–(37) yield $\varsigma_1 = C_1$, $\varsigma_2 = C_2$ and $\chi_1 = C_3$, $\chi_2 = C_4$. From equations (38) and (39), we obtain $\varsigma_1 = \varsigma_2 = 0$. Thus in this case we have ∂_f and ∂_g relative to the gauge functions $A_1 = x^2/2$, while $A_2 = 0$, respectively. Invoking (12) the two first integrals

$$\begin{aligned}
I_1 &= e^{f'}\cos(g') + e^{f'}(f'\cos g' - g'\sin(g')) - x^2/2, \\
I_2 &= e^{f'}\sin(g') + e^{f'}(f'\sin g' + g'\cos(g')),
\end{aligned} \tag{41}$$

of (30) can be deduced. Notice that the two operators ∂_f and ∂_g correspond to a single complex symmetry ∂_u . Also system (30) arises from the EL equation $(2 + u')u'' = xe^{-u'}$, with Lagrangian $L = u'e^{u'} + xu$.

3. Here we arrive at three Noether-like operators of the following systems of ODEs

$$\begin{aligned}
(2 + 4f' + f'^2 - g'^2)f'' - 2(2g' + f'g')g'' &= e^{-f'} \cos g', \\
(2 + 4f' + f'^2 - g'^2)g'' + 2(2g' + f'g')f'' &= -e^{-f'} \sin g',
\end{aligned} \tag{42}$$

that has a variational structure and admits Lagrangians

$$\begin{aligned}
L_1 &= (f'^2 - g'^2)e^{f'} \cos g' - 2f'g'e^{f'} \sin(g') + f, \\
L_2 &= 2f'g'e^{f'} \cos(g') + (f'^2 - g'^2)e^{f'} \sin(g') + g.
\end{aligned} \tag{43}$$

The system (42) admits the Noether-like operators ∂_x , ∂_f and ∂_g corresponding to the above Lagrangians. By using conditions (12) the four first integrals are found to be

$$\begin{aligned}
I_1 &= f - e^{f'}((f'^2 - g'^2) \cos g' - 2f'g' \sin g') - e^{f'}((f'^3 - 3f'g'^2) \cos g' - \\
&\quad (3f'^2g' - g'^3)) \sin g', \\
I_2 &= g - e^{f'}((f'^2 - g'^2) \sin g' + 2f'g' \cos g') - e^{f'}((f'^3 - 3f'g'^2) \sin g' + \\
&\quad (3f'^2g' - g'^3)) \cos g', \\
I_3 &= 2e^{f'}(f' \cos g' - g' \sin g') + e^{f'}((f'^2 - g'^2) \cos g' - 2f'g' \sin g') - x, \\
I_4 &= 2e^{f'}(f' \sin g' + g' \cos g') + e^{f'}((f'^2 - g'^2) \sin g' + 2f'g' \cos g').
\end{aligned} \tag{44}$$

of (42). Here, the first two invariants correspond to the Noether-like operator $\partial/\partial x$. In this case three operators can be transformed into a realization of the 2-dimensional Abelian complex algebra. The r-CODE here is $(2 + 4u' + u'^2)u'' = e^{-u'}$, which has Lagrangian $L = u'^2 e^{u'} + u$.

4. Now we look for a system which respects the second two-dimensional algebra in the complex domain. It helps us in constructing four Noether-like operators. For this we consider

$$\begin{aligned}
x(2 + f')f'' - xg'g'' &= f' + 1, \\
x(2 + f')g'' + xg'f'' &= g',
\end{aligned} \tag{45}$$

which admits four Noether-like operators

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial f}, \quad \mathbf{X}_2 = \frac{\partial}{\partial g}, \quad \mathbf{X}_3 = x \frac{\partial}{\partial x} + f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}, \\
\mathbf{X}_4 &= g \frac{\partial}{\partial f} - f \frac{\partial}{\partial g},
\end{aligned} \tag{46}$$

with respect to the Lagrangians

$$L_1 = \frac{e^{f'}}{x}(f' \cos(g') - g' \sin(g')), \quad L_2 = \frac{e^{f'}}{x}(g' \cos(g') + f' \sin(g')). \tag{47}$$

By utilizing (12) we determine the following four real first integrals

$$\begin{aligned}
I_1 &= ((1 + f') \cos(g') - g' \sin(g')) e^{f'}/x, \\
I_2 &= ((1 + f') \sin(g') + g' \cos(g')) e^{f'}/x, \\
I_3 &= ((f(1 + f') - gg' - x(f'^2 - g'^2)) \cos(g') - (g(1 + f') + \\
&\quad fg' - 2xf'g') \sin(g')) e^{f'}/x, \\
I_4 &= ((f(1 + f') - gg' - x(f'^2 - g'^2)) \sin(g') + (g(1 + f') + \\
&\quad fg' - 2xf'g') \cos(g')) e^{f'}/x.
\end{aligned} \tag{48}$$

for the system (45).

The above four Noether-like operators are also Noether symmetries of the system (45) as these satisfy (10) and (11). It would have been more difficult to determine these via the usual Noether approach especially \mathbf{X}_3 and \mathbf{X}_4 , which are mainly the result of complex encoding. This enters us into an open domain of nice and interesting problems in the complex domain with the clear indication that the solution of each or any of these would result in something remarkable in the real domain. Therefore, it is indispensable to classify such systems of ODEs with respect to the Noether-like operators they admit. The r-CODE is $x(2 + u')u'' = 1 + u'$, which has Lagrangian $L = u'e^{u'}/x$.

In [18], it is shown that the 5-dimensional Noether algebra of a free particle equation is a subalgebra of $sl(3, \mathbb{R})$ that give rise to five invariants. The Lagrangian that yields the maximum number of Noether symmetries is known as the standard Lagrangian. The concept of an alternative Lagrangian could be exercised to construct other invariants of differential equations. The invariants relative to such Lagrangians are expressed in terms of invariants of the standard Lagrangian. A similar situation arises in the case of systems of two ODEs. In the next example, we highlight the significance of an alternative Lagrangian in the variational problem of systems of ODEs.

Alternative Lagrangians:

The use of Lagrangians $L_1 = e^{f'} \cos(g') + f$, and $L_2 = e^{f'} \sin(g') + g$, in the EL-equations (4) yields the system

$$f'' = e^{-f'} \cos(g'), \quad g'' = -e^{-f'} \sin(g'). \tag{49}$$

These Lagrangians admit three Noether-like operators

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f}, \quad \mathbf{X}_3 = \frac{\partial}{\partial g}, \tag{50}$$

which provide the first integrals

$$\begin{aligned}
I_1 &= x - e^{f'} \cos(g'), \quad I_2 = e^{f'} \sin(g'), \\
I_3 &= e^{f'} (f' \cos(g') - g' \sin(g') - \cos(g')) - f, \\
I_4 &= e^{f'} (g' \cos(g') + f' \sin(g') - \sin(g')) - g,
\end{aligned} \tag{51}$$

for (49). The system (49) also has an alternative Lagrangians

$$\begin{aligned}
\tilde{L}_1 &= -\frac{1}{2x} \ln \left(1 + x^2 e^{-2f'} - 2x e^{-f'} \cos(g') \right), \\
\tilde{L}_2 &= \frac{-1}{x} \arctan \left(\frac{x e^{-f'} \sin(g')}{1 - x e^{-f'} \cos(g')} \right),
\end{aligned} \tag{52}$$

which possess the Noether-like operators

$$\mathbf{X}_4 = x \frac{\partial}{\partial x} + (x + f) \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}, \quad \mathbf{X}_5 = g \frac{\partial}{\partial f} - (x + f) \frac{\partial}{\partial g}. \tag{53}$$

The above operators reveal two more first integrals

$$\begin{aligned}
I_5 &= (1/2) \ln[(x - e^{f'} \cos(g'))^2 + e^{2f'} \sin^2 g'] + [(x - e^{f'} \cos(g')) \\
&\{e^{f'} (f' \cos(g') - g' \sin(g') - f' - f)\} - e^{f'} \sin(g') \{e^{f'} (g' \cos(g') + \\
&\quad f' \sin(g')) - g' - g\}] / ((x - e^{f'} \cos g')^2 + e^{2f'} \sin^2 g'), \\
I_6 &= \arctan(e^{f'} \sin(g') / e^{f'} \cos g' - x) + [(x - e^{f'} \cos(g')) \\
&\{e^{f'} (g' \cos(g') + f' \sin(g')) - g' - g\}] + e^{f'} \sin(g') \{e^{f'} (f' \cos(g') \\
&\quad - g' \sin g') - f' - f\} / ((x - e^{f'} \cos g')^2 + e^{2f'} \sin^2 g'),
\end{aligned} \tag{54}$$

for the system (49). It is important to see that alternative Lagrangians can also be used to reveal other first integrals of DEs. Perhaps more importantly from a physical point of view, alternative Lagrangians allow us to have a complete determination of all the physical constants or integrals (see [27]).

3 Classification of Noether-like operators

We firstly present the classification of three Noether point symmetries for complex Lagrangians on the line. Table 1 lists different cases of 3-dimensional Noether complex algebras together with their Lagrangians and representative EL r-CODEs. These are taken from [19].

We then classify systems of ODEs with respect to 5 and 6 Noether-like operators using CAS. We divide our discussion into two parts. There are three cases in which the commutators of Noether-like operators are closed which are

described in the first part. The other part contains the remaining cases. We obtain the classification of 5– and 6–dimensional Noether-like operators. These cases correspond to the cases described in Table 1. Moreover, in each case we write down the first integrals associated with the Noether-like operators which reveal an important feature of such operators.

Our purpose is to find out those systems of ODEs that can be mapped to r-CODEs via complex transformations. In particular, if a system of two ODEs with Lagrangians admitting at least 5 Noether-like operators is transformable to an r-CODE then the symmetry structure of that r-CODE will correspond to one of the cases described in Table 1. For instance, notice that the case $\mathcal{N}_{3,5}^1$ correspond to the first case in Table 1 as the Noether-like operators $\mathbf{X}_1, \dots, \mathbf{X}_5$ belong to complex symmetries ∂_x, ∂_u and $x\partial_x - u\partial_u$. Similarly, all cases described hereafter can be projected to each case in Table 1. This will help us in understanding the inverse problem for systems of two ODEs via r-CODEs.

Table 1. Classification of Noether Algebras for Lagrangians on the Line

No.	Noether symmetries	Lagrangians	Representative ELEs
$\mathcal{N}_{3,5}^1$	$\mathbf{Z}_1 = \partial_x, \mathbf{Z}_2 = \partial_u,$ $\mathbf{Z}_3 = x\partial_x - u\partial_u$	$L = -4u'^{1/2} + u$	$u'' = u'^{3/2}$
$\mathcal{N}_{3,5}^2$	$\mathbf{Z}_1 = \partial_x, \mathbf{Z}_2 = \partial_u,$ $\mathbf{Z}_3 = u\partial_x - x\partial_u$	$L = -(1 + u'^2)^{1/2} + xu'$	$u'' = (1 + u'^2)^{3/2}$
$\mathcal{N}_{3,6}^1$	$\mathbf{Z}_1 = \partial_u,$ $\mathbf{Z}_2 = x\partial_x + u\partial_u$ $\mathbf{Z}_3 = 2xu\partial_x + u^2\partial_u$	$L = \frac{u'}{x} + \frac{1}{2xu'}$	$xu'' = u'^3 - \frac{1}{2}u'$
$\mathcal{N}_{3,5}^3$	$\mathbf{Z}_1 = \partial_x, \mathbf{Z}_2 = \partial_u,$ $\mathbf{Z}_3 = (bx + u)\partial_x +$ $(bu - x)\partial_u$	$L = e^{-b \arctan u'} / (1 + b^2)$ $(1 + u'^2)^{1/2} + xu'$	$u'' = (1 + u'^2)^{\frac{3}{2}} e^{b \arctan u'}$
$\mathcal{N}_{3,6}^2$	$\mathbf{Z}_1 = \partial_u,$ $\mathbf{Z}_2 = x\partial_x + u\partial_u$ $\mathbf{Z}_3 = 2xu\partial_x +$ $(u^2 - x^2)\partial_u$	$L = \frac{\sqrt{1+u'^2}}{x} + \frac{Au'}{x}$	$xu'' = u'^3 + u' +$ $A(1 + u'^2)^{3/2}$

We now elaborate on the naming used. The indices in $\mathcal{N}_{i,j}^a$, are set as follows: a corresponds to different cases, i stands for the dimension of the complex algebras and j denotes the type of Noether-like operators that can arise from these cases. It may be noticed that we obtain four cases of 3–dimensional complex algebra that give rise to 5–Noether-like operators corresponding to

$\mathcal{N}_{3,5}^1$, $\mathcal{N}_{3,5}^2$, $\mathcal{N}_{3,5}^3$. Similarly, $\mathcal{N}_{3,6}^1$ and $\mathcal{N}_{3,6}^2$ are cases of 6–Noether-like operators. There is one more case $\mathcal{N}_{3,6}^3$ of 6–Noether-like operators corresponding to 3–dimensional complex Noether algebra. Due to its length, we have summarized it at the end of this section.

Closed Noether-like Operators:

Case $\mathcal{N}_{3,5}^1$

We first consider the system of nonlinear ODEs

$$\begin{aligned} f'' &= (f'^2 + g'^2)^{3/4} \cos\left(\frac{3}{2} \arctan\left(\frac{g'}{f'}\right)\right), \\ g'' &= (f'^2 + g'^2)^{3/4} \sin\left(\frac{3}{2} \arctan\left(\frac{g'}{f'}\right)\right), \end{aligned} \quad (55)$$

which can be obtained from the Lagrangians

$$\begin{aligned} L_1 &= -4(f'^2 + g'^2)^{1/4} \cos\left(\frac{1}{2} \arctan\left(\frac{g'}{f'}\right)\right) + f, \\ L_2 &= -4(f'^2 + g'^2)^{1/4} \sin\left(\frac{1}{2} \arctan\left(\frac{g'}{f'}\right)\right) + g, \end{aligned} \quad (56)$$

with the aid of (4). This system corresponds to the r-CODE listed for this case in Table 1. The system (55) admits the following five Noether-like operators

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f}, \quad \mathbf{X}_3 = \frac{\partial}{\partial g}, \quad \mathbf{X}_5 = -g \frac{\partial}{\partial f} + f \frac{\partial}{\partial g}, \\ \mathbf{X}_4 &= x \frac{\partial}{\partial x} - f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g}, \end{aligned} \quad (57)$$

corresponding to the above Lagrangians which can be verified from (6). Since the first three operators are translations in x, f and g they therefore leave both the system and the Lagrangians invariant. Consequently, these turn out to be Noether symmetries. The use of operator \mathbf{X}_1 in (12) gives

$$I_1 = 2r^{1/2} \cos(\theta/2) - f, \quad I_2 = 2r^{1/2} \sin(\theta/2) - g, \quad (58)$$

i.e., the two first integrals of the system (55) where r and θ are defined as

$$r^2 = f'^2 + g'^2, \quad \theta = \arctan(g'/f'). \quad (59)$$

In a similar way, $\partial/\partial f$ and $\partial/\partial g$ give the two invariants

$$I_3 = 2r^{-1/2} \cos(\theta/2) - x, \quad I_4 = 2r^{-1/2} \sin(\theta/2), \quad (60)$$

for the system (55). Similarly, for (55) one can determine the first integrals

$$\begin{aligned} I_5 &= 2r^{-1/2}(f \cos(\theta/2) + g \sin(\theta/2)) + xf - 2xr^{1/2} \cos(\theta/2) - 4, \\ I_6 &= 2r^{-1/2}(g \cos(\theta/2) - f \sin(\theta/2)) + xg - 2xr^{1/2} \sin(\theta/2), \end{aligned} \quad (61)$$

corresponding to \mathbf{X}_4 and \mathbf{X}_5 . We claim that any system exhibiting these operators can be mapped to the system (55). The commutators of these operators are

$$\begin{aligned} [\mathbf{X}_1, \mathbf{X}_2] &= [\mathbf{X}_2, \mathbf{X}_3] = [\mathbf{X}_1, \mathbf{X}_3] = [\mathbf{X}_1, \mathbf{X}_5] = [\mathbf{X}_4, \mathbf{X}_5] = 0, \\ [\mathbf{X}_1, \mathbf{X}_4] &= \mathbf{X}_1, \quad [\mathbf{X}_2, \mathbf{X}_5] = \mathbf{X}_3, \quad [\mathbf{X}_3, \mathbf{X}_5] = -\mathbf{X}_2, \\ [\mathbf{X}_2, \mathbf{X}_4] &= -\mathbf{X}_2, \quad [\mathbf{X}_3, \mathbf{X}_4] = -\mathbf{X}_3. \end{aligned} \quad (62)$$

Therefore, these form a closed algebra. Noteworthy, the system (55) admits the maximal 5-dimensional Noether-like operators.

Case $\mathcal{N}_{3,5}^2$

Consider the pair of two coupled Lagrangians

$$L_1 = -r_1^{1/2} \cos(\theta_1/2) + xf', \quad L_2 = -r_1^{1/2} \sin(\theta_1/2) + xg', \quad (63)$$

where

$$r_1^2 = (1 + f'^2 - g'^2)^2 + 4f'^2g'^2, \quad \theta_1 = \arctan\left(\frac{2f'g'}{1 + f'^2 - g'^2}\right). \quad (64)$$

The system that possesses these Lagrangians is

$$f'' = r_1^{3/2} \cos(3\theta_1/2), \quad g'' = r_1^{3/2} \sin(3\theta_1/2), \quad (65)$$

where r_1 and θ_1 are given by (64). This system corresponds to the r-CODE of Table 1. The five Noether-like operators admitted by the above system are

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f}, \quad \mathbf{X}_3 = \frac{\partial}{\partial g}, \\ \mathbf{X}_4 &= f \frac{\partial}{\partial x} - x \frac{\partial}{\partial f}, \quad \mathbf{X}_5 = g \frac{\partial}{\partial x} + x \frac{\partial}{\partial g}. \end{aligned} \quad (66)$$

The use of operators \mathbf{X}_2 and \mathbf{X}_3 in (12) gives rise to

$$\begin{aligned} I_1 &= x - r_1^{-1/2} (f' \cos(\theta_1/2) + g' \sin(\theta_1/2)), \\ I_2 &= r_1^{-1/2} (g' \cos(\theta_1/2) - f' \sin(\theta_1/2)). \end{aligned} \quad (67)$$

Now corresponding to $\partial/\partial x$, we have the first integrals

$$I_3 = r_1^{-1/2} \cos(\theta_1/2) + f, \quad I_4 = -r_1^{-1/2} \sin(\theta_1/2) + g, \quad (68)$$

for (65). By invocation of (12) the operators \mathbf{X}_4 and \mathbf{X}_5 give

$$\begin{aligned} I_5 &= \frac{x^2}{2} + \frac{f^2 - g^2}{2} + r_1^{-1/2} ((f - xf') \cos(\theta_1/2) + (xg' - g) \sin(\theta_1/2)), \\ I_6 &= fg + r_1^{-1/2} ((g - xg') \cos(\theta_1/2) - (f - xf') \sin(\theta_1/2)), \end{aligned} \quad (69)$$

We now calculate the commutators of these operators

$$\begin{aligned} [\mathbf{X}_1, \mathbf{X}_2] &= [\mathbf{X}_2, \mathbf{X}_3] = [\mathbf{X}_1, \mathbf{X}_3] = [\mathbf{X}_3, \mathbf{X}_4] = [\mathbf{X}_2, \mathbf{X}_5] = 0, \\ [\mathbf{X}_1, \mathbf{X}_6] &= 0, \quad [\mathbf{X}_1, \mathbf{X}_4] = -\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_4] = \mathbf{X}_1, \quad [\mathbf{X}_3, \mathbf{X}_5] = \mathbf{X}_1, \\ [\mathbf{X}_2, \mathbf{X}_6] &= \mathbf{X}_3, \quad [\mathbf{X}_3, \mathbf{X}_6] = 2\mathbf{X}_2, \quad [\mathbf{X}_4, \mathbf{X}_6] = -\mathbf{X}_5, \\ [\mathbf{X}_5, \mathbf{X}_6] &= -\mathbf{X}_4, \quad [\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_3, \quad [\mathbf{X}_4, \mathbf{X}_5] = f\partial_g + g\partial_f = \mathbf{X}_6, \end{aligned} \quad (70)$$

Notice that we obtain an extra operator \mathbf{X}_6 which is not a Noether-like operator. The nice feature about it is that it forms a closed algebra together with other 5 Noether-like operators.

Case $\mathcal{N}_{3,6}^1$

We obtain six Noether-like operators by using (6)

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial f}, \quad \mathbf{X}_2 = \frac{\partial}{\partial g}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial x} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \\ \mathbf{X}_4 &= g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}, \quad \mathbf{X}_5 = 2xf\frac{\partial}{\partial x} + (f^2 - g^2)\frac{\partial}{\partial f} + 2fg\frac{\partial}{\partial g}, \\ \mathbf{X}_6 &= 2xg\frac{\partial}{\partial x} + 2fg\frac{\partial}{\partial f} - (f^2 - g^2)\frac{\partial}{\partial g}, \end{aligned} \quad (71)$$

for the following system of ODEs

$$\begin{aligned} xf'' &= -f'^3 + 3f'g'^2 - f'/2, \\ xg'' &= -3f'^2g' + g'^3 - g'/2, \end{aligned} \quad (72)$$

which arise from the r-CODE in Table 1. By the utilization of (4) it can be verified that the above system admits the Lagrangians

$$L_1 = \frac{f'}{x} - \frac{f'}{2x(f'^2 + g'^2)}, \quad L_2 = \frac{g'}{x} + \frac{g'}{2x(f'^2 + g'^2)}. \quad (73)$$

Our next step is to construct first integrals for the system (72) by taking into account all Noether-like operators. Now starting with the Noether-like operators \mathbf{X}_1 and \mathbf{X}_2 we arrive at the two real first integrals

$$I_1 = \frac{1}{x} + \frac{f'^2 - g'^2}{2x(f'^2 + g'^2)^2}, \quad I_2 = \frac{f'g'}{x(f'^2 + g'^2)^2}. \quad (74)$$

The other first integrals corresponding to \mathbf{X}_3 and \mathbf{X}_4 are

$$\begin{aligned} I_3 &= -\frac{f'}{f^2 + g^2} + \frac{f}{x} + \frac{f(f'^2 - g'^2) + 2gf'g'}{2x(f'^2 + g'^2)^2}, \\ I_4 &= \frac{g'}{f^2 + g^2} + \frac{g}{x} + \frac{g(f'^2 - g'^2) - 2ff'g'}{2x(f'^2 + g'^2)^2}. \end{aligned} \quad (75)$$

Similarly, \mathbf{X}_5 and \mathbf{X}_6 yield

$$\begin{aligned} I_5 &= 2x - \frac{(f^2 - g^2)}{x} + \frac{2(ff' + gg')}{f'^2 + g'^2} + \frac{(f^2 - g^2)(f'^2 - g'^2) + 4ff'gg'}{2x(f'^2 + g'^2)^2}, \\ I_6 &= \frac{2fg}{x} - \frac{2(f'g - fg')}{f'^2 + g'^2} + \frac{fg(f'^2 - g'^2) - f'g'(f^2 - g^2)}{x(f'^2 + g'^2)^2}. \end{aligned} \quad (76)$$

The closed algebra of these Noether-like operators is spanned by

$$\begin{aligned} [\mathbf{X}_1, \mathbf{X}_2] &= [\mathbf{X}_3, \mathbf{X}_4] = [\mathbf{X}_5, \mathbf{X}_6] = 0, \\ [\mathbf{X}_1, \mathbf{X}_3] &= \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_4] = -\mathbf{X}_2, \quad [\mathbf{X}_1, \mathbf{X}_5] = 2\mathbf{X}_3, \\ [\mathbf{X}_1, \mathbf{X}_6] &= 2\mathbf{X}_4, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_4] = \mathbf{X}_1, \\ [\mathbf{X}_2, \mathbf{X}_5] &= -2\mathbf{X}_4, \quad [\mathbf{X}_2, \mathbf{X}_6] = 2\mathbf{X}_3, \quad [\mathbf{X}_3, \mathbf{X}_5] = \mathbf{X}_5, \\ [\mathbf{X}_3, \mathbf{X}_6] &= \mathbf{X}_6, \quad [\mathbf{X}_4, \mathbf{X}_5] = \mathbf{X}_6, \quad [\mathbf{X}_4, \mathbf{X}_6] = -\mathbf{X}_5. \end{aligned} \quad (77)$$

Indeed these operators span a six-dimensional algebra.

Remaining Cases:

Case $\mathcal{N}_{3,5}^3$

Consider the pair of two real Lagrangians

$$\begin{aligned} L_1 &= \frac{-r^{1/2}}{(1+b^2)} e^{\frac{b}{2}\theta_1} (\cos(\theta/2)) + xf', \\ L_2 &= \frac{-r^{1/2}}{(1+b^2)} e^{\frac{b}{2}\theta_1} (\sin(\theta/2)) + xg'. \end{aligned} \quad (78)$$

The system of second-order ODEs associated with the above Lagrangians is

$$f'' = r^{3/2} e^{-\frac{b}{2}\theta_1} (\cos(3\theta/2)), \quad g'' = r^{3/2} e^{-\frac{b}{2}\theta_1} (\sin(3\theta/2)), \quad (79)$$

where r and θ are given by (59) and θ_1 by (64). These correspond to the r-CODE in Table 1. The three Noether-like operators admitted by (79) are

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f}, \quad \mathbf{X}_3 = \frac{\partial}{\partial g}. \quad (80)$$

The first integrals are

$$\begin{aligned} I_1 &= x + \frac{r^{-1/2}}{(1+b^2)} e^{\frac{b}{2}\theta_1} ((b-f') \cos(\theta/2) - g' \sin(\theta/2)), \\ I_2 &= -\frac{r^{-1/2}}{(1+b^2)} e^{\frac{b}{2}\theta_1} ((b-f') \sin(\theta/2) + g' \cos(\theta/2)), \end{aligned} \quad (81)$$

for the system (79) associated with \mathbf{X}_2 and \mathbf{X}_3 . Now corresponding to \mathbf{X}_1 we have

$$\begin{aligned} I_3 &= \frac{r^{-1/2}}{(1+b^2)} e^{\frac{b}{2}\theta_1} ((1+bf') \cos(\theta/2) + bg' \sin(\theta/2)) + f, \\ I_4 &= \frac{r^{-1/2}}{(1+b^2)} e^{\frac{b}{2}\theta_1} (bg' \cos(\theta/2) - (1+bf') \sin(\theta/2)) + g. \end{aligned} \quad (82)$$

We can obtain two more first integrals corresponding to other Noether operators. These operators correspond to alternate Lagrangians. These integrals can be expressed as a combination of above four first integrals.

Case $\mathcal{N}_{3,6}^2$

Consider the system of two second-order ODEs

$$\begin{aligned} x f'' &= f^3 - 3f'g'^2 + f' + r_1^{3/2} (A \cos(3\theta_1/2)), \\ x g'' &= 3f'^2 g' - g'^3 + g' + r_1^{3/2} (A \sin(3\theta_1/2)), \end{aligned} \quad (83)$$

where r_1 and θ_1 are given by (64). This system arises from the r-CODE in Table 1. The Lagrangians of the above system are readily found to be of the forms

$$L_1 = \frac{r_1^{1/2}}{x} \cos(\theta_1/2) + \frac{A f'}{x}, \quad L_2 = \frac{r_1^{1/2}}{x} \sin(\theta_1/2) + \frac{A g'}{x}. \quad (84)$$

The six Noether-like operators related to (83) are shown to be

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial f}, & \mathbf{X}_2 &= \frac{\partial}{\partial g}, & \mathbf{X}_3 &= x \frac{\partial}{\partial x} + f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}, \\ \mathbf{X}_4 &= g \frac{\partial}{\partial f} - f \frac{\partial}{\partial g}, & \mathbf{X}_5 &= 2xf \frac{\partial}{\partial x} + (f^2 - g^2 - x^2) \frac{\partial}{\partial f} + 2fg \frac{\partial}{\partial g}, \\ \mathbf{X}_6 &= 2xg \frac{\partial}{\partial x} + 2fg \frac{\partial}{\partial f} - (f^2 - g^2 - x^2) \frac{\partial}{\partial g}. \end{aligned} \quad (85)$$

We take \mathbf{X}_1 and \mathbf{X}_2 which imply the first integrals

$$\begin{aligned} I_1 &= \frac{A}{x} + \frac{r_1^{-1/2}}{x} (f' \cos(\theta_1/2) + g' \sin(\theta_1/2)), \\ I_2 &= \frac{1}{x} r_1^{-1/2} (f' \sin(\theta_1/2) - g' \cos(\theta_1/2)), \end{aligned} \quad (86)$$

for (83). Other Noether-like operators \mathbf{X}_3 and \mathbf{X}_4 enables us to write the first integrals

$$\begin{aligned} I_3 &= r_1^{-1/2} \left(\left(1 + \frac{ff' - gg'}{x} \right) \cos \frac{\theta_1}{2} + \frac{(f'g + fg')}{x} \sin \frac{\theta_1}{2} \right) + \frac{Af}{x}, \\ I_4 &= r_1^{-1/2} \left(\frac{(f'g + fg')}{x} \cos \frac{\theta_1}{2} - \left(1 + \frac{ff' - gg'}{x} \right) \sin \frac{\theta_1}{2} \right) + \frac{Ag}{x}, \end{aligned} \quad (87)$$

The remaining two invariants related to the Noether-like operators $\mathbf{X}_5, \mathbf{X}_6$ are found to be

$$\begin{aligned} I_5 &= A \left(x + \frac{f^2 - g^2}{x} \right) - r_1^{-1/2} \left(xf' - \frac{(f^2 - g^2)f' - 2fgg'}{x} \right) \cos \frac{\theta_1}{2} \\ &\quad - r_1^{-1/2} \left(xg' - 2f - 2g + \frac{(f^2 - g^2)g' + 2fgf'}{x} \right) \sin \frac{\theta_1}{2}, \\ I_6 &= \frac{2Afg}{x} + r_1^{-1/2} \left(xf' - \frac{f'(f^2 - g^2) - 2fgg'}{x} \right) \sin \frac{\theta_1}{2} \\ &\quad + r_1^{-1/2} \left(xg' - 2f - 2g - \frac{g'(f^2 - g^2) + 2f'fg}{x} \right) \cos \frac{\theta_1}{2}. \end{aligned} \quad (88)$$

Case $\mathcal{N}_{3,6}^3$

Lastly, a nonlinear system of ODEs can be mapped to the the r-CODE

$$u'' = A \left(\frac{1 + u'^2 + (u - xu')^2}{1 + x^2 + u^2} \right)^{3/2}, \quad (89)$$

which admits the 3-dimensional complex Noether algebra $\mathbf{Z}_1 = (1 + x^2)\partial_x + xu\partial_u$, $\mathbf{Z}_2 = xu\partial_x + (1 + u^2)\partial_u$, $\mathbf{Z}_3 = u\partial_x - x\partial_u$. By choosing \mathbf{Z}_1 , we have the complex first integral

$$I_1 = A \frac{\alpha}{\sqrt{1 + \alpha^2}} + \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}}, \quad (90)$$

where $\alpha = u(1 + x^2)^{-1/2}$, $\beta = u'(1 + x^2)^{1/2} - xu(1 + x^2)^{-1/2}$. The complex Lagrangian of the above r-CODE is

$$L = \frac{1}{t^2}(1 + x^2)^{-3/2} ((\gamma - u') \sin \theta + \delta \sec \theta) + d(x, u)u' + e(x, u), \quad (91)$$

where d and e satisfy $d_x = e_u + A(1 + x^2 + u^2)^{-3/2}$ and $t = \sqrt{1 + x^2 + u^2}/(1 + x^2)$, $\delta = tu$, $\tan \theta = (u' - \gamma)/t$ and $\gamma = xu/(x^2 + 1)$. The other two complex first integrals corresponding to the respective \mathbf{Z}_2 and \mathbf{Z}_3 are given as

$$\begin{aligned} I_2 &= \frac{Ax}{\sqrt{1 + x^2 + u^2}} - \frac{u'}{\sqrt{1 + u'^2 + (u - xu')^2}}, \\ I_3 &= \frac{A}{\sqrt{1 + x^2 + u^2}} + \frac{(x^2 + 1)^{-3/2}}{t^2} (\delta \sin \theta + \delta \sec \theta), \end{aligned} \quad (92)$$

where t , γ , δ and θ are given above and $\delta = x + u\gamma$. The complex Lagrangian, Noether symmetries and first integrals exactly correspond to the emerging system of (89).

4 Maximal Noether algebra for two-dimensional systems

For many problems of physical interest, linear systems as well as some non-linear systems, the Lagrangians that arise do result in the maximum number of Noether point symmetries for the said system, are known as standard Lagrangians. These Lagrangians are constructed in a straightforward and obvious manner from physical considerations. A Lagrangian of the given dynamical equation is said to be maximally symmetric if it gives the maximum number of Noether symmetries that can occur for the dynamical equation. We know that for a dynamical equation, infinitely many Lagrangians may exist, but regarding one Lagrangian, there is a unique dynamical equation. The symmetries of Lagrangian admitted by the systems of ODEs is of great significance especially in regard to their use in physical applications. Each Noether symmetry for the given system provides one constant of motion. The analytic continuation furnishes two nice invariant quantities for the corresponding system of

ODEs. The conservation laws play a vital role in the study of physical phenomena. As an example it would be significant to mention that if a dynamical system remains invariant under translation in time then it has a conservation of energy. Likewise, a rotational symmetry correspond to the conservation of angular momentum. The complex symmetries correspond to conformal motions. Therefore, if a differential equation and its Lagrangian admit a complex rescaling symmetry then that dynamical system respects conservation of infinitesimal angles.

We now focus our attention on a two-dimensional system of free particle equations. The determination of Noether symmetries and all invariants for this simplest system

$$f'' = 0, \quad g'' = 0, \quad (93)$$

can be obtained via the classical Noether approach. We utilize the complex approach here. The maximal nine Noether-like operators for (93) relative to the Lagrangians $L_1 = \frac{1}{2}(f'^2 - g'^2)$ and $L_2 = f'g'$ are found to be (note that these are not standard)

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x}, & \mathbf{X}_2 &= \frac{\partial}{\partial f}, & \mathbf{X}_3 &= \frac{\partial}{\partial g}, & \mathbf{X}_4 &= 2x\frac{\partial}{\partial x} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \\ \mathbf{X}_5 &= g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}, & \mathbf{X}_6 &= x\frac{\partial}{\partial f}, & \mathbf{X}_7 &= x\frac{\partial}{\partial g}, \\ \mathbf{X}_8 &= x^2\frac{\partial}{\partial x} + x(f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}), & \mathbf{X}_9 &= x(g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}). \end{aligned} \quad (94)$$

Notice that the operators $\mathbf{X}_1, \dots, \mathbf{X}_8$ are the usual Noether symmetries with respect to the standard Lagrangian of (93). Surprisingly, the maximal 8-dimensional Noether algebra (excluding \mathbf{X}_9) for the above system can be obtained from the analytic continuation of the maximal 5-dimensional Noether algebra, i.e.,

$$\begin{aligned} \mathbf{Z}_1 &= \frac{\partial}{\partial x}, & \mathbf{Z}_2 &= \frac{\partial}{\partial u}, & \mathbf{Z}_3 &= 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, & \mathbf{Z}_5 &= x\frac{\partial}{\partial u}, \\ \mathbf{Z}_4 &= x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \end{aligned} \quad (95)$$

for the equation $u'' = 0$, relative to the Lagrangian $L = u'^2/2$, (termed as the standard Lagrangian since it is the kinetic energy of the particle) which is maximally symmetric since it yields the maximum number of Noether point symmetries (95). The extra operator \mathbf{X}_9 seems to be a symmetry corresponding to \mathbf{Z}_4 . It does not satisfy the classical Lie conditions and it thus fails to be a symmetry of (93). Yet it *yields* first integrals, see $I_{4,1}$ and $I_{4,2}$, in Table 2. Hence, this surplus symmetry is of an extraordinary characteristic. It is,

somehow, attached to the system in a very strange way, although it is not a Lie symmetry. Indeed this extra ordinary feature of complex variables plays a central role in the determination of invariants of two-dimensional systems thereby yields deeper insights into the analysis of physical models ([28]).

Table 2: First Integrals for simplest two dimensional system

Symmetries	Complex First Integrals	First Integrals
∂_x	$I_1 = u'^2$	$I_{1,1} = f'^2 - g'^2$ $I_{1,2} = f'g'$
∂_u	$I_2 = u'$	$I_{2,1} = f'$ $I_{2,2} = g'$
$x\partial_u$	$I_3 = xu' - u$	$I_{3,1} = xf' - f$ $I_{3,2} = xg' - g$
$2x\partial_x + u\partial_u$	$I_4 = -xu'^2 + uu'$	$I_{4,1} = -x(f'^2 - g'^2) + ff' - gg'$ $I_{4,2} = -xf'g' + fg' + f'g$
$x^2\partial_x + xu\partial_u$	$I_5 = xuu' - \frac{x^2u'^2}{2} - \frac{u^2}{2}$	$I_{5,1} = x(ff' - gg') - \frac{x^2}{2}(f'^2 - g'^2) -$ $\frac{1}{2}(f^2 - g^2)$ $I_{5,2} = x(fg' + f'g) - x^2f'g' - fg$

We know that for any particle Lagrangian, the maximum dimension of the Noether algebra is five [18]. As mentioned earlier, the free particle equation admits another Lagrangian of the type $L = u^2u' \log u' - u^2u' - xuu'^2 + \frac{1}{6}x^2u'^3 + au' + b$, where a and b are related by $a_x = b_u$. The Noether symmetries of this Lagrangian are

$$\mathbf{Z}_6 = u \frac{\partial}{\partial x}, \quad \mathbf{Z}_7 = x \frac{\partial}{\partial x}, \quad \mathbf{Z}_8 = xu \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial u}. \quad (96)$$

The Noether symmetries of the two Lagrangians together give the complete symmetry generators of $sl(3, \mathbb{C})$ for the restricted particle equation. In the solution of simple harmonic oscillator the two Lagrangian pictures are essential in order to describe fully the algebra and periodicity of motion of the oscillator [27]. The complete algebraic description, i.e., the complete determination of all the physical constants is obtained from these. We also know that the simplest system admits $sl(4, \mathbb{R})$. The question of complete determination of invariants of scalar linear equations can be extended to system of two free particle equations and it can be asked whether one can find the rest of the first integrals

corresponding to a 7-dimensional Noether subalgebra of $sl(4, \mathbb{R})$. Therefore, we present Conjecture 1.

Conjecture 1. *The 7-dimensional subalgebra of $sl(4, \mathbb{R})$ for the simplest system which complements the 8-dimensional Noether algebra is derivable from a complex subalgebra of 5-dimension of an r-CODE via an alternative Lagrangian.*

It is important to mention that the use of alternative Lagrangians grants more Noether-like operators which would determine complete set of physical constants for the simplest system. We know that every linear second-order r-CODE

$$u'' = \alpha(x)u' + \beta(x)u + \gamma(x), \quad (97)$$

is equivalent to the simplest r-CODE, $U'' = 0$ via an appropriate point transformation [1]. It is also known that it admits a 5-dimensional Noether algebra with respect to a standard Lagrangian. Thus, for systems of linear ODEs which can be mapped to a linear r-CODE the Noether-like operators may induce an 8-dimensional Noether algebra. Consequently, we state the Conjecture 2.

Conjecture 2. *Every system of two linear second-order ODEs that is obtainable from a linear r-CODE with usual Lagrangian admits the 8-dimensional Noether algebra.*

5 Conclusion

In this paper, we have addressed the problem of Noether classification for systems of two second-order EL ODEs that arise from the submaximal and maximal Noether symmetry classification of Lagrangians on the line. We achieved this by the introduction of Noether-like operators. In this way, we provided an algebraic study of systems of ODEs which are variational and obtainable from complex EL equations. Also this study provides new ways of obtaining first integrals for those classes of systems of nonlinear equations that have not been studied before via the classical Noether approach. It is certainly meaningful to classify all such systems which exhibit the structure of Noether-like operators. We have discussed these for five, six and eight Noether-like operators. In some cases the Noether-like operators turn out to be the Noether point symmetries. The Noether counting theorem, an analogue of the Lie counting theorem, states that a second-order EL r-CODE can have 0, 1, 2, 3 or 5-dimensional Noether algebra [18]. We have shown that a system of two second-order ODEs that arise from an r-CODE can have 0, 1, 2, 3, 4, 5, 6 or 9-dimensional Noether-like operators. The lower dimensional cases up to dimension 4 were illustrated by means of examples. The simplest system possesses the maximal 8-dimensional Noether point symmetry algebra which

we achieved remarkably by the analytic continuation of the 5-dimensional Noether algebra of a linear second-order r-CODE. In addition it supplies an extra operator which is not a Noether symmetry of the system, although it does permit the emergence of a first integral.

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